

Stability of Topological Insulators with Non-Abelian Edge Excitations

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Abstract

Chiral-antichiral pairs of non-Abelian Hall states, like the Pfaffian, Read-Rezayi and NASS states, can be used to model two-dimensional time-reversal invariant topological insulators. Their stability was shown to be associated to the presence of a \mathbb{Z}_2 anomaly and characterized by the same \mathbb{Z}_2 index introduced for free fermion and Abelian systems. In this work, we continue the stability analysis by providing the form of time-reversal invariant interactions that gap the non-Abelian edge excitations. Our approach is based on the description of non-Abelian states as projections of corresponding “parent” Abelian states.

1 Introduction

Topological states of matter in two and three dimensions are currently investigated both theoretically [1] and experimentally [2] [3], and new systems have been suggested in combined two-three dimensional geometries [4]. Of particular interest are the time-reversal invariant systems, such as the topological insulators, that can be realized in absence of external magnetic fields. They are characterized by gapful excitations in the bulk and massless charged edge excitations.

Two-dimensional topological insulators with interacting fermions, i.e. different from band insulators, can be modeled by pairs of quantum Hall states carrying opposite spin and chirality, such that time-reversal invariant systems are obtained [5]. Assuming the existence of a suitable bulk Hamiltonian that gives rise to both sets of edge excitations, the question remains of the interactions between electrons of opposite chirality at the edge, that can gap completely the system and made it topologically trivial. Some interactions may be forbidden by time-reversal symmetry, possibly combined with other discrete symmetries, leading to a so-called symmetry protected topological phase [6]. Thus, the stability analysis is particularly relevant in this kind of theoretical modeling [5] [7] [8].

In our earlier paper [9], a stability criterium based on symmetry arguments [5] was generalized to time-reversal invariant topological insulators with non-Abelian edge excitations (see also [10]). In the present work, we complement this analysis with the study of the allowed interactions. Analogous analyses are currently developed for interacting two-dimensional topological superconductors [11] and for two-dimensional systems on the surface of three-dimensional topological insulators [4] [12] [13].

1.1 Flux insertion argument and edge interactions

The stability analysis of topological insulators was based on two approaches: the first used symmetry and topology arguments and the Kramers theorem; the second involved a systematic study of edge interactions that do not break time-reversal symmetry. In the first approach, pioneered by Fu, Kane and Mele [14] [15] [16], one considers the response of the system, say in the annulus geometry, to the insertion of flux quanta at the center – a refinement of the well-known Laughlin argument for the quantized Hall current [17] [18]. For example, in the case of a single spinful free fermion mode, the addition of half flux $\Phi_0/2$ let the ground state $|\Omega\rangle$ evolve into a spin $S = 1/2$ neutral edge excitation, corresponding to a change of the \mathbb{Z}_2 index

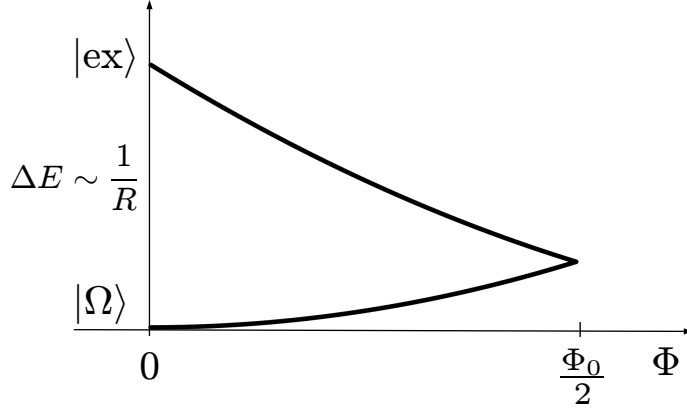


Figure 1: Kramers degeneracy at half flux quantum.

$(-1)^{2S}$:

$$\Phi = 0 : (-1)^{2S} = 1 \quad \longrightarrow \quad \Phi = \frac{\Phi_0}{2} : (-1)^{2S} = -1. \quad (1.1)$$

In time-reversal invariant systems, the Kramers theorem implies that this edge state is degenerate with another $S = 1/2$ state $|\text{ex}\rangle$ coming from the excitations (see Fig.1). This degeneracy is robust with respect to any perturbation that preserves time-reversal symmetry, and it implies that ground state and excitations at $\Phi = 0$ are separated by a gap $\Delta E = O(1/R)$, that vanishes in the large volume limit. Therefore, the time-reversal invariant topological insulator of one free fermion has gapless edge excitations and is topologically non trivial. The argument clearly extends to N free fermion systems that are stable (resp. unstable) for N odd (even).

The flux insertion argument was generalized by Levin and Stern [5] to systems with Abelian edge excitations, that can be described by multicomponent Luttinger liquids [19]. The \mathbb{Z}_2 index was shown to extend as follows:

$$(-1)^{2\Delta S}, \quad 2\Delta S = \frac{\nu^\uparrow}{e^*}, \quad (1.2)$$

where ν^\uparrow is the Hall filling fraction and e^* is the minimal fractional charge, in units of e , of one chiral component (equivalently, the dimensionless ratio of the spin Hall conductivity and minimal charge σ_{sH}/e^*). In a system with anyonic excitations, the ratio (1.2) correctly measures the spin ΔS of the smallest electron excitation created

at the edge by adding half fluxes. An odd (resp. even) ratio corresponds to stable (unstable) systems, generalizing the number of fermion modes in the non-interacting case.

The second stability analysis was based on studying the possible time-reversal invariant electron interactions at the edge [5] [7]. Their general expressions in multi-component Abelian systems can be written using vertex operators of the bosonic conformal field theory in the so-called K -matrix formalism. It was found in Ref. [5] that the stability of the edge modes is again determined by the index (1.2): when this is positive, i.e. the Kramers degeneracy argument does not hold, there are enough interactions for gapping all edge modes; conversely, if the index is negative, i.e. there is Kramers degeneracy, one mode remains gapless. In conclusion, both approaches of studying flux insertions and interactions led to the same conclusions about the stability of topological insulators with Abelian edges.

1.2 Non-Abelian topological insulators

In a recent paper [9], we were able to extend the first kind of stability analysis to topological insulators involving pairs of non-Abelian edge excitations, like the Pfaffian [20], the Read-Rezayi [21] and the non-Abelian Spin Singlet States [22]. We first obtained the grand-canonical partition function of the conformal field theory describing chiral-antichiral pairs of edge excitations, by dwelling on earlier results [23] [24]. Then, we used the partition function for discussing the flux argument in full generality for any interacting system.

Quantum Hall edge states possess neutral modes that can be Abelian or non-Abelian. The corresponding conformal theories have the affine symmetry $U(1) \times G/H$, where $U(1)$ is the charge symmetry and G is another (non-Abelian) symmetry characterizing the neutral part (possibly a coset G/H) [24]. The electron field is represented by the product of a chiral vertex operator $V = e^{i\alpha\varphi}$ for the charge part and the neutral field ψ_e of the G/H theory:

$$\Psi_e = e^{i\alpha\varphi} \psi_e . \quad (1.3)$$

Even in non-Abelian theories, the field ψ_e should have Abelian fusion rules with all fields in the theory, i.e. should be a so-called simple current [25]. This field can be used to build the partition function on the spacetime torus that is modular invariant, i.e. symmetric under discrete coordinate changes that respect the double periodicity [25]. It was found that the partition function of any quantum Hall state is uniquely

determined by two inputs: the choice of neutral G/H theory and of Abelian field ψ_e in the theory that represents the neutral part of the electron [24].

The study of transformations of the partition function under flux insertions [9] showed that the neutral sectors do not play any role, and that the stability depends on a pair of numbers (k, p) parameterizing the charge spectrum, namely the value of the minimal charge, $e^* = 1/p$, and of the would-be chiral Hall conductivity (spin conductivity), $\nu^\uparrow = k/p$. The minimal spin ΔS excitation created by flux insertions was found to match the Levin-Stern index (1.2), again

$$2\Delta S = \frac{\nu^\uparrow}{e^*} = k, \quad (-1)^{2\Delta S} = (-1)^k. \quad (1.4)$$

In particular, the simpler topological insulator made by a pair of Pfaffian states is unstable, since it corresponds to $e^* = 1/4$ and $\nu^\uparrow = 1/2$, i.e. to $(k, p) = (2, 4)$.

In the same paper, we emphasized that the change of index under flux insertions expresses a discrete \mathbb{Z}_2 anomaly, that is the non-conservation of the (edge) spin parity $(-1)^{2S}$ of the fermionic system [14] [10] [26]. Indeed, this anomaly is the remnant of the $U(1)_S$ continuous anomaly of the spin Hall effect after the inclusion of relativistic corrections, such as spin-orbit interaction, that break the spin conservation explicitly (but keep time-reversal invariance) [1].

The behaviour of the partition function under modular transformations was also analyzed [9]. In presence of fermionic excitations, this function always possesses four parts (spin sectors), called the Neveu-Schwarz and Ramond sectors and their tildes; these are characterized by antiperiodic and periodic boundary conditions for the fermion field in each direction [25]. It was found that the half-flux insertions and the modular transformations S and T map the four spin sectors among themselves. In the case of unstable non-anomalous systems, time-reversal symmetry allows to sum the four sectors together, leading to the complete modular invariant Z_{Ising} ; on the contrary, in the presence of \mathbb{Z}_2 anomaly they cannot be summed up, leading to four independent partition functions, as follows:

$$\begin{aligned} Z_{\text{Ising}} &= Z^{NS} + Z^{\widetilde{NS}} + Z^R + Z^{\widetilde{R}}, & k \text{ even, unstable,} \\ Z_{\text{TR}} &= \left(Z^{NS}, Z^{\widetilde{NS}}, Z^R, Z^{\widetilde{R}} \right), & k \text{ odd, stable.} \end{aligned} \quad (1.5)$$

Therefore, the \mathbb{Z}_2 spin parity anomaly was associated to a discrete gravitational anomaly, i.e. to a lack of complete modular invariance of the torus partition function [27].

1.3 Interactions in non-Abelian topological insulators

In our previous paper [9] we did not discuss the allowed edge interactions, thus we could not check whether a non-anomalous system, not protected by the Kramers theorem, does actually become fully gapped. In this paper, we provide the expressions of a sufficient set of gapping interactions passing all physics tests.

We use the known result that some non-Abelian states can be described as projections of corresponding “parent” Abelian states [28][29]. For example, the (331) Halperin state of distinguishable electrons is related to the Pfaffian state by the projection onto states of identical electrons, e.g. by antisymmetrizing over wavefunction coordinates. Since this projection does not affect the time-reversal symmetry of states and operators, it maps time-reversal invariant interactions between the two theories. Clearly, the value of the Levin-Stern index is equal in the unprojected (Abelian) and projected (non-Abelian) theories [9]. Through this mapping the earlier study of Abelian edge interactions [5] [7] [8] can be extended to non-Abelian cases, where it provides the possible electrons interactions. It turns out that these are not sufficient to completely gap the system, but further neutral quasiparticle interactions can be introduced that do this task when the Levin-Stern index is one.

The plan of the paper is the following. In Section two, we recall the analysis of time-reversal interactions in multicomponent Abelian theories. In Section three, we introduce and use the projection from the Abelian (331) state into the Pfaffian to obtain the non-Abelian interactions. In analyzing their properties, special attention is paid to normal ordering of fields and to the absence of spontaneous breaking of time-reversal symmetry. In Section four, we generalize this results to the \mathbb{Z}_k -parafermionic Read-Rezayi states. In Section five, we similarly study topological insulators made by pairs of non-Abelian spin-singlet (NASS) states, which turn out to be all unstable. Finally, in Section six we present our conclusions.

2 Time-reversal invariant interactions in Abelian theories

We start by recalling the conformal field theory description of bosonic fields within the K -matrix formalism [19]. In the case of topological insulators, the lattice of excitations is doubled to account for the pairs of fields with opposite spin and chirality, as described in [5] [7] (we adopt the notation and conventions of [7]). We introduce

the $2N \times 2N$ symmetric invertible matrix \mathcal{K} with integer components: time-reversal symmetry determines its form to be,

$$\mathcal{K} = \begin{pmatrix} K & W \\ W^T & -K \end{pmatrix}, \quad (2.1)$$

where K is the usual symmetric $N \times N$ matrix of Abelian Hall systems and $W^T = -W$.

A generic (multi)-electron excitation is specified by a vector Λ with $2N$ integer components, such that its statistics, $\theta/\pi = \Lambda^T \mathcal{K} \Lambda$, and charge, $Q = \Lambda^T \rho$, are integer-valued, where ρ is the so-called charge vector. In our basis, this is made of two equal N -dimensional vectors, $\rho = (\rho^\uparrow, \rho^\downarrow)$, $\rho^\uparrow = \rho^\downarrow = (1, \dots, 1)$; then, the elementary electron excitations correspond to the basis vectors $\Lambda = e_i$, that are equal to one in the i -th position and zero elsewhere, $i = 1, \dots, 2N$. The electrons are represented by normal ordered vertex operators of the $2N$ -component bosonic field $\Phi(t, x)$, as follows:

$$\Psi_i^\dagger(t, x) =: \exp(-ie_i^T \mathcal{K} \Phi(t, x)) :, \quad i = 1, \dots, 2N. \quad (2.2)$$

If $W = 0$ in (2.1), the first N operators, $i = 1, \dots, N$, represent chiral spin-up electrons and the second N ones antichiral spin-down electrons; if $W \neq 0$, the first (resp. second) N operators describe electrons with spin up (down) with mixed chiralities.

The time-reversal \mathcal{T} transformations act on the bosonic field as follows [5]:

$$\mathcal{T} \Phi(t, x) \mathcal{T}^{-1} = \Sigma_1 \Phi(-t, x) + \pi \mathcal{K}^{-1} \Sigma_\downarrow \rho, \quad (2.3)$$

where

$$\Sigma_1 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \Sigma_\downarrow = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad (2.4)$$

are $2N \times 2N$ block matrices. Time-reversal symmetry implies $\mathcal{K} = -\Sigma_1 \mathcal{K} \Sigma_1$, and $\rho = \Sigma_1 \rho$.

The time-reversal transformations of the basic fermionic fields (2.2) are as follows, keeping in mind that \mathcal{T} is antiunitary,

$$\mathcal{T} : \quad \Psi_i^\dagger =: \exp(-ie_i^T \mathcal{K} \Phi) : \rightarrow : \exp(-i(\Sigma_1 e_i)^T \mathcal{K} \Phi - i\pi e_i^T \Sigma_\downarrow \rho) :, \quad (2.5)$$

namely

$$\mathcal{T} : \quad \Psi_i^\dagger \rightarrow \Psi_{i+N}^\dagger, \quad \Psi_{i+N}^\dagger \rightarrow -\Psi_i^\dagger, \quad i = 1, \dots, N. \quad (2.6)$$

2.1 Time-reversal symmetric interactions

The Hamiltonian H_{int} of electronic edge interactions is expressed in terms of vertex operators U_{Λ_i} as follows:

$$H_{\text{int}} = \int dt \sum_i g_i U_{\Lambda_i} + \text{h.c.},$$

$$U_{\Lambda_i}(t, x) =: \exp \left(-i \Lambda_i^T \mathcal{K} \Phi(t, x) \right) :, \quad (2.7)$$

where Λ_i are integer vectors subjected to the conditions specified below. The coupling constant g_i can be complex and space dependent to account for interactions at impurities, possibly leading to $g_i \rightarrow \infty$, such that both relevant and irrelevant interactions should be considered.

The condition obeyed by the Λ_i for admissible interactions are [5] [7]:

i) charge neutrality,

$$Q = \Lambda_i^T \rho = 0; \quad (2.8)$$

ii) mutual locality of all interactions (Haldane null vector criterion [30]),

$$\frac{\theta}{\pi} = \Lambda_i^T \mathcal{K} \Lambda_j = 0, \quad \forall i, j, \quad (2.9)$$

such that each interaction can effectively freeze one Abelian edge mode;

iii) time-reversal invariance of H_{int} ,

$$\Sigma_1 \Lambda_i = \pm \Lambda_i, \quad \Lambda_i^T \Sigma_{\downarrow} \rho = \text{even}, \quad \forall i, \quad (2.10)$$

as obtained from (2.3);

iv) linear independence of the Λ_i and, more strongly, the following minimality, or “primitivity”, condition [5],

$$n_1 \Lambda_1 + \cdots + n_k \Lambda_k \neq m \Lambda, \quad \text{with } m > 1. \quad (2.11)$$

Solutions to this equation with integer Λ vector and $m > 1$ could imply spontaneous symmetry breaking of time-reversal symmetry. For example, in the one component case, the square of the mass term $U = (\bar{\Psi}^\dagger \Psi)^2$ is time-reversal invariant, but it would induce the time-reversal breaking expectation value $\langle \bar{\Psi}^\dagger \Psi \rangle \neq 0$.

The stability analysis of Refs. [5] [7] answered the question of whether there exist enough interactions Λ_i for gapping all N modes in a given Abelian theory, thus leading to a trivial massive phase. It was found that there always exist $(N - 1)$ gapping

interactions for any matrix \mathcal{K} , such that one massless mode is possible, at most. Furthermore the N -th gapping interaction was show to be time-reversal invariant when the \mathbb{Z}_2 Levin-Stern index is one, thus matching the results of the flux insertion argument. Let us recall the main steps of this analysis.

The $(N-1)$ solutions Λ_i of conditions (2.8)-(2.11) are eigenvectors of the Σ_1 matrix (2.4) with eigenvalue one, that can be taken of the form:

$$\Lambda_1 = (\Lambda_1^\uparrow, \Lambda_1^\downarrow) = (\underbrace{1, -1, 0, \dots, 0}_N, \underbrace{1, -1, 0, \dots, 0}_N), \quad (2.12)$$

\vdots

$$\Lambda_{N-1} = (\Lambda_{N-1}^\uparrow, \Lambda_{N-1}^\downarrow) = (\underbrace{1, 0, \dots, 0, -1}_N, \underbrace{1, 0, \dots, 0, -1}_N). \quad (2.13)$$

These vectors are globally neutral, $Q = \Lambda_i^T \rho = 0$, but they also have neutral chiral and antichiral components, $\Lambda_i^{\uparrow T} \rho^\uparrow = \Lambda_i^{\downarrow T} \rho^\downarrow = 0$. As shown in Refs. [5] [7], these vectors also satisfy the other conditions, (2.9)-(2.11), irrespectively of the form of \mathcal{K} (2.1).

On the other hand, the N -th solution depends explicitly on \mathcal{K} ; for simplicity, we shall consider the diagonal form for \mathcal{K} , i.e. with $W = 0$. We define the vector:

$$\bar{\Lambda} = r \begin{pmatrix} K^{-1} \rho^\uparrow \\ -K^{-1} \rho^\downarrow \end{pmatrix}, \quad (2.14)$$

where r is the smallest integer such that all components of $\bar{\Lambda}$ are integers. This vector is (necessarily) a eigenvector of Σ_1 with eigenvalue -1 ; it obeys all the conditions (2.8)-(2.11), but the integer quantity,

$$R = -\bar{\Lambda}^T \Sigma_\downarrow \rho = r \rho^{\downarrow T} K^{-1} \rho^\downarrow, \quad (2.15)$$

should be even for a time-reversal invariant interaction (cf. Eq.(2.10)). It turns out that R is equal to the Levin-Stern quantity $2\Delta S = \nu^\uparrow/e^*$, characterizing the flux-insertion stability approach discussed in the Introduction. The proof of this equivalence requires some assumptions on the form of the K matrix [5]; in the cases of interest in this paper, we use the facts that K is definite positive and invariant under permutations of rows. Then, the fractional charge of chiral excitations, defined by $Q = n^T K^{-1} \rho^\uparrow$, with n a N -dimensional integer vectors, take minimal value e^* for $n = (1, 0, \dots, 0)$, and this value is equal to the inverse of r in (2.15). Since the filling fraction is given by $\nu^\uparrow = \rho^{\uparrow T} K^{-1} \rho^\uparrow$, we finally obtain $R = 2\Delta S$.

3 Time-reversal interactions in the Pfaffian topological insulator

3.1 From the (331) state to the Pfaffian state

The Pfaffian state is the simplest non-Abelian quantum Hall state [20]; from the values of filling fraction and minimal charge, we determine its stability index (M is odd integer):

$$\nu^\uparrow = \frac{1}{M+1}, \quad e^* = \frac{1}{2M+2}, \quad 2\Delta S = \frac{\nu^\uparrow}{e^*} = 2, \quad (-1)^{2\Delta S} = 1. \quad (3.1)$$

Therefore, the Pfaffian topological insulator is unstable. In the following, its gapping interactions will be deduced from those of the “parent” Abelian topological state, the (331) state defined by the K -matrix:

$$K = \begin{pmatrix} 2+M & M \\ M & 2+M \end{pmatrix}. \quad (3.2)$$

(For general $M = 1, 3, \dots$, we should rather call it the $(M+2, M+2, M)$ state).

The correspondence between (331) and Pfaffian (chiral) Hall states is rather well established, and follows from the way the two systems describe electrons forming bosonic pairs [29]. From the K matrix (3.2) we read the expression of the (331) ground state wavefunction:

$$\Psi_{(331)}(z_i; w_j) = \prod_{i < j}^N z_{ij}^{2+M} \prod_{i < j}^N w_{ij}^{2+M} \prod_{i,j}^N (z_i - w_j)^M, \quad (3.3)$$

where $z_{ij} = z_i - z_j$, $w_{ij} = w_i - w_j$. The two sets of coordinates z_i and w_i , $i = 1, \dots, N$, pertain to electrons that are distinct by an additional quantum number, say isospin up and down, and the wavefunction is antisymmetric under exchanges of coordinates of the same kind. Therefore, in the bosonic case $M = 0$, this function does not vanish when two electrons of opposite isospin meet at the same point. This feature signals the pairing of distinct electrons.

The Pfaffian state also describes electrons forming pairs with parallel spin, (p -wave superconductor) and its wavefunction has the same vanishing property. However, in this case all the electrons are identical and the wavefunction is completely antisymmetric. Its expression can be obtained from the (331) wavefunction by antisymmetrizing with respect to all $2N$ electron coordinates, such that the isospin quantum number

is washed out. Indeed, the following relation holds [29]:

$$\Psi_{\text{Pfaff}}(z_i, z_{i+n}) = \mathcal{A} [\Psi_{(331)}(z_i; w_j)] = \prod_{i < j}^{2N} z_{ij}^{M+1} \text{Pf} \left(\frac{1}{z_i - z_j} \right), \quad (3.4)$$

where $\mathcal{A}[\dots]$ denotes antisymmetrization over all the $2N$ coordinates.

The projection to identical fermions corresponds to a map between the conformal field theory descriptions of the two states. In order to describe this correspondence, we should first separate the neutral and charged components of electron fields (cf. (2.2)). In the (331) theory, both parts are expressed by vertex operators of chiral bosonic fields, φ and ϕ , that are linear combinations of earlier field Φ (chiral part):

$$V = \exp(i\alpha\varphi), \quad F = \exp(i\phi), \quad (3.5)$$

with $\alpha = \sqrt{M+1}$. The dimensions of the fields are $h = \alpha^2/2 = (M+1)/2$ and $h = 1/2$, respectively. The field F is actually a Weyl fermion whose charge does not contribute to Q but accounts for the isospin. Thus, the charge-neutral decomposition reads:

$$\Psi_1 = V F, \quad \Psi_2 = V F^\dagger, \quad (3.6)$$

where Ψ_i have been defined earlier within the K -matrix lattice.

It is instructive to write the (331) wavefunction as a correlator of the bosonic fields V and F :

$$\begin{aligned} \Psi_{(331)} &= \langle V(z_1) \cdots V(z_N) V(w_1) \cdots V(w_N) \rangle \langle F(z_1) \cdots F(z_N) F^\dagger(w_1) \cdots F^\dagger(w_N) \rangle \\ &= \left(\prod_{i < j} z_{ij} w_{ij} \prod_{i,j} (z_i - w_j) \right)^{M+1} \det \left(\frac{1}{z_i - w_j} \right). \end{aligned} \quad (3.7)$$

This expression is actually equal to (3.3) owing to the Cauchy determinant identity [25] (up to an overall constant).

The projection from the Abelian to the Pfaffian states is obtained by identifying the two species of Abelian fermions $\Psi_1 \sim \Psi_2$, thus eliminating the isospin quantum number [29]. This amounts to projecting the Weyl fermion to a neutral Majorana fermion, $F \rightarrow \chi$ and $F^\dagger \rightarrow \chi$, namely:

$$\Psi_1 \rightarrow V \chi, \quad \Psi_2 \rightarrow V \chi. \quad (3.8)$$

After this replacement, the wavefunction (3.7) is now modified in the second term, involving the correlator of $2N$ Majorana fields, that produces the Pfaffian expression in (3.4). The clustering property of this wavefunction is reproduced by the fusion

rule of Majorana fields $\chi \cdot \chi \sim I$. The corresponding conformal theory changes from the Abelian $U(1) \times U(1)$ to the non-Abelian $U(1) \times \text{Ising}$ theory, with central charges $c = 2$ and $c = 3/2$, respectively [20].

We now describe the corresponding map between the Abelian and Pfaffian topological insulators. The Abelian insulator is defined by the \mathcal{K} matrix (2.1) in block-diagonal form ($W = 0$), and K given by (3.2). Besides the chiral spin-up Hall states discussed so far, there are corresponding antichiral spin-down states, whose electrons fields Ψ_i , $i = 3, 4$ in (2.2) are similarly projected into antichiral Pfaffian fields: $\Psi_{3,4} \rightarrow \bar{V} \bar{\chi}$. Summarizing, we have the following map between electrons in the two theories ($i = 1, 2$),

$$\begin{aligned} \Psi_i^\dagger &= : \exp(-ie_i^T \mathcal{K} \Phi) : & \rightarrow & : \exp(-i\alpha\varphi) : \chi = V^\dagger \chi, \\ \Psi_i &= : \exp(ie_i^T \mathcal{K} \Phi) : & \rightarrow & : \exp(i\alpha\varphi) : \chi = V \chi, \\ \Psi_{i+2}^\dagger &= : \exp(-i\bar{e}_i^T \mathcal{K} \Phi) : & \rightarrow & : \exp(i\alpha\bar{\varphi}) : \bar{\chi} = \bar{V}^\dagger \bar{\chi}, \\ \Psi_{i+2} &= : \exp(i\bar{e}_i^T \mathcal{K} \Phi) : & \rightarrow & : \exp(-i\alpha\bar{\varphi}) : \bar{\chi} = \bar{V} \bar{\chi}. \end{aligned} \quad (3.9)$$

In this table, we also explain our notation for conformal fields: the bar denotes antichirality, e.g. $\varphi = \varphi(z)$, $\bar{\varphi} = \bar{\varphi}(\bar{z})$, while the dagger refers to Fock space operators.

The time-reversal transformations of electron fields (2.6) are left invariant by the projection, and act on the the electrons of the Pfaffian theory as follows:

$$\begin{aligned} \mathcal{T} : \quad \Psi_i^\dagger &= V^\dagger \chi & \rightarrow & \Psi_{i+2}^\dagger = \bar{V}^\dagger \bar{\chi}, & i = 1, 2, \\ \Psi_{i+2}^\dagger &= \bar{V}^\dagger \bar{\chi} & \rightarrow & -\Psi_i^\dagger = -V^\dagger \chi. \end{aligned} \quad (3.10)$$

It would be tempting to assign the minus sign to the transformation of the Majorana field χ and leave invariant the charged Abelian field V , up to antiunitarity. However, this choice is not correct and actually could not be consistently applied to the parafermions of Read-Rezayi states to be discussed in Section 4.2. The correct choice is:

$$\begin{aligned} \mathcal{T} : \quad V^\dagger &\rightarrow \bar{V}^\dagger, & \bar{V}^\dagger &\rightarrow -V^\dagger, \\ \chi &\rightarrow \bar{\chi}, & \bar{\chi} &\rightarrow \chi. \end{aligned} \quad (3.11)$$

These transformation rules are motivated by the following argument. The sign is due to the 2π spin rotation of $(2+1)$ -dimensional spinors and is not built-in in the $(1+1)$ -dimensional conformal theory description. Nevertheless, it should hinge on a conserved quantity of the conformal theory. According to the flux-insertion argument discussed in the Introduction, spin is represented in this theory by the difference of

charges for chiral-spin-up and antichiral-spin-down conformal fields; thus, the fermion number sign can be written:

$$(-1)^{N_F} = (-1)^{2S} = (-1)^{Q-\bar{Q}}. \quad (3.12)$$

Therefore, it is a property to be assigned to the charged fields, and the neutral fields (Abelian and non-Abelian) transform as scalar quantities. Note that this action of time-reversal transformations on edge fermions is specific of topological insulators and is different from that of topological superconductors, whose neutral Majorana fields should carry the fermion number necessarily [1] [16] [27].

3.2 Projected interactions

The two time-reversal-invariant interactions of the (331) Abelian theory are obtained by the methods described in Section two; the first one is associated to the lattice vector Λ_1 (2.12) and the second one is obtained by specializing the expression of the vector $\bar{\Lambda}$ (2.14) for the K matrix (3.2). They read:

$$\Lambda_1 = (1, -1, 1, -1), \quad \bar{\Lambda} = (1, 1, -1, -1). \quad (3.13)$$

These vectors determine the following normal-ordered product of fermionic fields (cf. (2.7)):

$$\begin{aligned} U_{\Lambda_1} &=: \Psi_1^\dagger \Psi_2 \Psi_3^\dagger \Psi_4 : + \text{h.c.}, \\ U_{\bar{\Lambda}} &=: \Psi_1^\dagger \Psi_2^\dagger \Psi_3 \Psi_4 : + \text{h.c.}, \end{aligned} \quad (3.14)$$

(recall that the $\Psi_{1,2}$ are chiral spin-up and $\Psi_{3,4}$ are antichiral spin-down). We now apply the projection to these expressions for obtaining time-reversal invariant interactions in the Pfaffian topological insulator. Since the maps (3.9) apply to individual fermion fields, we should first undo the normal ordering in (3.14) by point splitting, then apply the projection and finally re-normal order the result in the Pfaffian theory. Let us consider the two interactions U_{Λ_1} and $U_{\bar{\Lambda}}$ in turn.

We use the normal-ordering formula for vertex operators [25],

$$: \exp(i\alpha\varphi(z_1)) : : \exp(i\beta\varphi(z_2)) := (z_{12})^{\alpha\beta} : \exp(i\alpha\varphi(z_1) + i\beta\varphi(z_2)) :, \quad (3.15)$$

to rewrite:

$$U_{\Lambda_1} = \lim_{z_1 \rightarrow z_2} z_{12}^M \bar{z}_{12}^M \Psi_1^\dagger(z_1) \Psi_2(z_2) \Psi_3^\dagger(\bar{z}_1) \Psi_4(\bar{z}_2) + \text{h.c.} \quad (3.16)$$

We now perform the projection (3.9), for each field in this expression and then use (3.15) to normal-order the vertex operators VV^\dagger again. We obtain:

$$U_{\Lambda_1} \rightarrow U_{\Lambda_1}^{\text{Pfaff}} = \lim_{z_1 \rightarrow z_2} \left[\frac{1}{z_{12}} : V^\dagger(z_1) V(z_2) : \chi(z_1) \chi(z_2) \right]_{\text{reg.}} \times \left[z \rightarrow \bar{z} \right]. \quad (3.17)$$

Next, we consider the product expansions of chiral vertex operators and of chiral Majorana fields for $z_1 \rightarrow z_2$, focusing on the chiral parts. These expansions involve descendant fields in the conformal representation (sector) of the identity field I of both the charged $c = 1$ and neutral Majorana $c = 1/2$ theories, owing to the fusion rules $V^\dagger \cdot V \sim I$ and $\chi \cdot \chi = I$: schematically, $U_{\Lambda_1}^{\text{Pfaff}} = [I]_{c=1} [I]_{c=1/2}$ [25]. Upon using (3.15), we find the following terms in the charged part (omitting constants),

$$: V^\dagger(\varepsilon) V(0) := 1 + \varepsilon \partial \varphi + \varepsilon^2 ((\partial \varphi)^2 + \partial^2 \varphi) + O(\varepsilon^3), \quad (3.18)$$

and in the neutral part,

$$\chi(\varepsilon) \chi(0) = \frac{1}{\varepsilon} + \varepsilon : \chi \partial \chi : + O(\varepsilon^3). \quad (3.19)$$

The expression of $U_{\Lambda_1}^{\text{Pfaff}}$ is obtained by selecting the finite terms for $\varepsilon \rightarrow 0$ in the product of (3.18) and (3.19). This is the normal-ordering procedure for general conformal theories to be further discussed later. The final expression for the interaction is obtained as follows, neglecting total derivatives:

$$U_{\Lambda_1}^{\text{Pfaff}} = (2T_n + \alpha^2 T_c)(2\bar{T}_n + \alpha^2 \bar{T}_c), \quad \alpha^2 = M + 1, \quad (3.20)$$

where $T_n = -\chi \partial \chi / 2$ and $T_c = -(\partial \varphi)^2 / 2$ are the stress tensors of the Majorana fermion and bosonic theory, respectively [25].

In the case of the interaction $U_{\bar{\Lambda}}$ in (3.14), we follow similar steps and arrive to the expression:

$$U_{\bar{\Lambda}} \rightarrow U_{\bar{\Lambda}}^{\text{Pfaff}} = \lim_{z_1 \rightarrow z_2} \left[z_{12} : V(z_1)^\dagger V(z_2)^\dagger : \chi(z_1) \chi(z_2) \right]_{\text{reg.}} \times \left[z \rightarrow \bar{z} \right] + \text{h.c.} \quad (3.21)$$

The conformal sectors involved are $U_{\bar{\Lambda}}^{\text{Pfaff}} = [V^{\dagger 2}]_{c=1} [I]_{c=1/2}$, since the original Abelian interaction had charged chiral/antichiral parts, i.e. $Q = (2, -2)$ in the notation of Section two. After re-normal ordering, we finally obtain:

$$U_{\bar{\Lambda}}^{\text{Pfaff}} =: (V^\dagger)^2 (\bar{V})^2 : + \text{h.c.} \quad (3.22)$$

Let us discuss the normal-ordering procedure employed in deriving the non-Abelian interactions. At first sight, one could notice a certain degree of arbitrariness in choosing the (singular) term to be extracted from the Laurent expansions in ε (3.18),

(3.19): actually, there is none. Each interaction is associated to a specific representation (sector) of the conformal theory, respectively $[I] \times [I]$ and $[V^{\dagger 2}] \times [I]$, that is selected by the quantum numbers of the electrons involved. The operator expansions are series with coefficients the descendant fields in these representations [25]. The normal-ordering procedure identifies the first significant term in the series, made of primary or descendant (quasi-primary) fields. Had we chosen another term of the series, its ability for gapping the system would have been the same, since higher descendants in the representation obey the same selection rules for coupling.

In non-Abelian theories, operator expansions may involve more than one sector, leading to several interaction channels, that could be selected by different normal orderings. However, in our fermionic interactions there is always a single channel, because the fermions are Abelian fields, as discussed in the Introduction.

3.3 Properties of non-Abelian interactions

We discuss some features of the interactions found in the non-Abelian theory and their ability to gap the system. The two edge interactions U_{Λ_1} and $U_{\bar{\Lambda}}$ just obtained are time-reversal invariant because they involve bosonic stress tensors $T\bar{T}$ and squares of the V vertex operator, respectively.

The expression of $U_{\Lambda_1}^{\text{Pfaff}}$ is quartic in the Majorana field; actually, the simpler quadratic interaction in the (331) Abelian theory [5], $U = \Psi_1^\dagger \Psi_4 - \Psi_2^\dagger \Psi_3 + \text{h.c.}$, can be shown to vanish after projection into the Pfaffian. Thus, the quartic interaction $U_{\Lambda_1}^{\text{Pfaff}}$ should be considered as minimal, or “primitive” (cf. (2.11)).

The second interaction $U_{\bar{\Lambda}}^{\text{Pfaff}}$ is also quartic, with V transforming under time-reversal as a single fermion in the one-component Abelian theory. As discussed in Section two, the analogous Abelian interaction $(\Psi^\dagger \Psi)^2$ must be discarded, because it would imply the symmetry breaking expectation value of the “square-root” $\langle \Psi^\dagger \Psi \rangle \neq 0$ [5] [7]. In the Pfaffian theory the corresponding time-reversal breaking quantity $V^\dagger \bar{V}$ cannot acquire an expectation value because it is not local with respect to some excitations.

For example, we consider the chiral quasiparticle $V_\sigma \sigma$ of minimal charge $Q = 1/4$ (for $M = 1$), that can exist at the edge when a corresponding quasiparticle is present in the bulk. The operator product expansion between the charge part $V_\sigma = \exp(i1/\sqrt{8}\varphi)$ and $V^\dagger \bar{V}$ is given by $V_\sigma V^\dagger \sim z^{-1/2}$. Therefore, the following

correlator involving two σ has a square-root branch-cut and nontrivial monodromy:

$$\begin{aligned} \langle V^\dagger(0) \bar{V}(0) V_\sigma(z e^{i2\pi}) V_\sigma(w) \rangle &= - \langle V^\dagger(0) \bar{V}(0) V_\sigma(z) V_\sigma(w) \rangle \\ &\neq \langle V^\dagger(0) \bar{V}(0) \rangle \langle V_\sigma(z) V_\sigma(w) \rangle. \end{aligned} \quad (3.23)$$

Equation (3.23) implies $\langle V^\dagger \bar{V} \rangle = 0$. On the contrary, the Abelian operator $\Psi^\dagger \Psi$ can acquire an expectation value because is local with respect to all excitations.

Next, we discuss the effect of interactions $U_{\Lambda_1}^{\text{Pfaff}}$ (3.20) and $U_{\Lambda}^{\text{Pfaff}}$ (3.22) within perturbation expansion. Let us consider correlation functions of fermionic fields $\psi = V \chi$ and $\bar{\psi} = \bar{V} \bar{\chi}$ to a given perturbative order of the interaction $g U_{\Lambda}$:

$$\langle \psi^\dagger(z_1) \cdots \psi(z_i) \cdots \bar{\psi}^\dagger(\bar{z}_j) \cdots \bar{\psi}(\bar{z}_N) g \int U_{\Lambda} \cdots g \int U_{\Lambda} \rangle_{CFT}. \quad (3.24)$$

Using fusion rules and operator product expansions, we can check whether these expressions are non-vanishing, thus breaking conformal invariance and eventually gapping the spectrum.

Let us first test this formula for the N -component Abelian theory of Section two. In this case, the fusion of the interaction specified by Λ_k in (2.12) with the basic chiral and antichiral fermions, respectively $\Psi_i = \Psi_i(z)$ and $\bar{\Psi}_{i+n} = \bar{\Psi}_i(\bar{z})$, $i = 1, \dots, N$, takes the following form (for simplicity, consider $\mathcal{K} = \text{diag}(K, -K)$):

$$\Psi_i^\dagger(z) U_{\Lambda_k}(0) \sim z^{e_i^T K \Lambda_k^\dagger}, \quad \bar{\Psi}_i^\dagger(\bar{z}) U_{\Lambda_k}(0) \sim \bar{z}^{\bar{e}_i^T K \Lambda_k^\dagger}. \quad (3.25)$$

In presence of $n < N$ interactions Λ_k , we can find $(N - n)$ vectors e_i (resp. \bar{e}_i) for which the expression at the exponent of z (resp. \bar{z}) vanishes, since K is an invertible matrix: the electrons corresponding to these e_i and \bar{e}_i do not couple to the interactions and remain massless. Therefore, exactly $n = N$ perturbations Λ_k are needed to break scale invariance of the entire spectrum. The argument can be repeated for quasiparticle excitations.

In Abelian theories, we further observe that the cosine interactions (2.7) localize the values of bosonic fields at a minimum, leading to gapful excitations — the so-called Haldane's semiclassical gapping criterium [5] [7] [30]. Therefore, the stronger statement can be made that the N interactions not only break scale invariance but give mass to all excitations.

In the Pfaffian theory, we first distinguish between interactions that do or do not involve charge transfer between chiralities. Specifically, the interaction $U_{\Lambda}^{\text{Pfaff}}$, with chiral and antichiral charges $(Q, \bar{Q}) = (2, 2)$, is non-vanishing for the correlator,

$$\langle \psi^\dagger(z_1) \psi^\dagger(z_2) \bar{\psi}(\bar{z}_3) \bar{\psi}(\bar{z}_4) U_{\Lambda}^{\text{Pfaff}} \rangle \neq 0. \quad (3.26)$$

that describes scattering with charge transfer between the two chiralities. This process can be realized by charged excitations, electrons and quasiparticles, that all acquire mass by the Haldane argument.

3.3.1 Gapping neutral excitations

The Pfaffian theory further possesses neutral quasiparticle excitations that do not couple to $U_{\Lambda}^{\text{Pfaff}}$. Indeed, their scattering processes do not involve charge transfer and correspond to vanishing chiral correlators, as e.g. $\langle \psi^\dagger(z_1) \psi(z_2) U_{\Lambda}^{\text{Pfaff}} \rangle = 0$. Neutral excitations are affected by the other interaction $U_{\Lambda}^{\text{Pfaff}}$ with $(Q, \bar{Q}) = (0, 0)$, since $\langle \psi^\dagger(z_1) \psi(z_2) U_{\Lambda}^{\text{Pfaff}} \rangle \neq 0$. In the following, we show that this coupling breaks scale invariance but cannot give mass to neutral excitations.

The single-edge partition function provides the list of quasiparticle sectors of the Pfaffian topological insulator. As described in our earlier paper, this is obtained by summing the Neveu-Schwarz and Ramond sectors, $NS, \widetilde{NS}, R, \widetilde{R}$, whose expressions are given in Appendix A.4 [9], leading to:

$$\begin{aligned} Z_{\text{Pf TI}} &= Z_{NS} + Z_{\widetilde{NS}} + Z_R + Z_{\widetilde{R}} \\ &= 2 \sum_{a=-3}^4 \left(|K_a I|^2 + |K_a \psi|^2 + |K_a \sigma|^2 \right). \end{aligned} \quad (3.27)$$

In this expression, the K_a are characters of $U(1)$ representations corresponding to the Abelian parts of excitations, carrying charge $Q = a/4 + 2\mathbb{Z}$, while the characters I , ψ and σ describe the neutral non-Abelian parts, being the identity, fermion and spin of the Ising model, respectively [25].

In presence of the charged interaction $U_{\Lambda}^{\text{Pfaff}}$ with large coupling, all charged excitations become highly massive, such that $K_a \rightarrow \delta_{a,0}$ in (3.27) (up to an irrelevant factor). Therefore, there remain the neutral excitations of the Ising model,

$$Z_{\text{Pf TI}} \rightarrow Z_{\text{Ising}} = |I|^2 + |\psi|^2 + |\sigma|^2, \quad (3.28)$$

that are time-reversal invariant and non-chiral.

The consequences of adding the interaction $g U_{\Lambda}^{\text{Pfaff}} = g T_n \bar{T}_n$ to the Ising model, with $T_n \sim \chi \partial \chi$ its stress tensor, are well understood in the literature [31]. This irrelevant interaction can be used to describe the renormalization group flow from the tricritical to the critical Ising model, as view from the low-energy end point. All along this flow, the Majorana field χ stays massless, thus this interaction does not gap the system. A direct way to check this fact is to notice that the derivative field

present in T implies a power-law correction to the single-particle dispersion relation, $\varepsilon(k) \sim v|k| + g|k|^3$. Thus, values of $g \neq 0$ break conformal invariance but leave a massless neutral state at low-energy. This should be contrasted with the results in the (331) parent Abelian theory, where the neutral interaction is also of cosine type and yields massive excitations. Note also that a $T\bar{T}$ interaction is possible in any CFT consistently with time-reversal symmetry: it would imply power-law correlations and power-law localization of neutral modes that are irrelevant.

We now remark that the Ising model (3.28) possesses the relevant interaction,

$$U_{pq} = m\bar{\chi}\chi, \quad (3.29)$$

that generically gives mass to the theory, unless fine-tuning to $m = 0$ is considered. This corresponds to a quasiparticle interaction in the original Pfaffian topological insulator, those possibility was not considered before. Actually, earlier discussions were limited to electron interactions, corresponding to impurity scattering, that are local with respect to all chiral excitations. Quasiparticle interactions, such as $m\bar{\chi}\chi$, were discarded because they can be non-local with some chiral quasiparticles. However, in the reduced theory (3.28), electrons and charged chiral quasiparticles have disappeared, thus the $m\bar{\chi}\chi$ interaction is local with respect to the remaining neutral (non-chiral) excitations and is acceptable.

In conclusion, in the Pfaffian topological insulator we introduced a quasiparticle interaction for gapping the neutral non-Abelian modes that is allowed when the charged excitations are infinitely massive. This argument requires a separation of scales between heavy charged excitations and light neutral excitations, that is not necessary in Abelian systems. Moreover, such quasiparticle interaction is generically unavoidable, but the underlying physical mechanism for its occurrence is yet unclear. We can only speculate that it is a sort of gravitational interaction at the edge.

We finally remark that quasiparticle interactions could be possible in all theories, Abelian and non-Abelian, provided the charged modes first acquire mass. The stability analysis has indeed shown that this is the crucial fact, governed by the Levin-Stern index, and determines all other results.

4 Time-reversal interactions in the Read-Rezayi topological insulators

4.1 Projected interactions

The Read-Rezayi Hall states describe the binding of identical electrons in clusters of k elements, extending the earlier $k = 2$ case of the Pfaffian [21]. The clustering implies that the ground state (bosonic) wavefunction does not vanish when k coordinates coincide. In the conformal field theory description, the electrons are represented by the \mathbb{Z}_k parafermion field χ_1 , whose k -th fusion with itself yields the identity, $(\chi_1)^k \sim I$, leading to a non-vanishing correlator at coincident points. The parafermion conformal field theory is denoted by PF_k and can be realized by the coset $PF_k = SU(2)_k/U(1)$ [24]. As usual, excitations also have a charge part expressed by vertex operators, leading to the theory $U(1) \times PF_k$, with central charge $c = 1 + c_k$, $c_k = 2(k-1)/(k+2)$. The fusion of $n < k$ parafermions χ_1 define the parafermion field χ_n , and these fields obey Abelian fusion rules among themselves:

$$\chi_i \cdot \chi_j \sim \chi_\ell, \quad \ell = i + j \bmod k, \quad (4.1)$$

that conserve a \mathbb{Z}_k quantum number. Moreover, they obey $\chi_n^\dagger = \chi_{k-n}$.

The parent Abelian theory is a k -fluid generalization of the (331) state with the following K matrix [21] [29]:

$$K_{ij} = \begin{cases} M + 2 & i = j = 1, \dots, k, \\ M & i \neq j. \end{cases} \quad (4.2)$$

This theory describes the clustering of distinguishable electrons, since its wavefunction does not vanish (for $M = 0$) when k electrons of different species meet at the same point. This wavefunction reproduce the Read-Rezayi expression upon complete antisymmetrization with respect to all coordinates, (see Ref.[29] for a complete discussion of the projection). The two systems share the same spectrum of charges: the filling fraction and minimal charge are,

$$\nu^\uparrow = \frac{k}{kM + 2}, \quad e^* = \frac{1}{kM + 2}, \quad 2\Delta S = k, \quad (-1)^{2\Delta S} = (-1)^k. \quad (4.3)$$

Thus, the flux insertion argument tells that the topological insulators made by pairs of Read-Rezayi states are stable (unstable) for k odd (even). The stability parameters are $(k, p) = (k, kM + 2)$.

We again use the projection from the Abelian states to determine the gapping interaction of the Read-Rezayi topological insulators. In the present case the projection maps k different chiral species into a single one and the corresponding electron fields Ψ_i , $i = 1, \dots, k$ in (2.2) are projected into the Read-Rezayi electron $\psi = V\chi_1$. More precisely, the correspondence is as follows ($i = 1, \dots, k$):

$$\begin{aligned}\Psi_i^\dagger &=: \exp(-ie_i^T \mathcal{K}\Phi) : & \rightarrow & : \exp(-i\alpha\varphi) : \chi_1 = V^\dagger \chi_1, \\ \Psi_i &=: \exp(ie_i^T \mathcal{K}\Phi) : & \rightarrow & : \exp(i\alpha\varphi) : \chi_1^\dagger = V \chi_{k-1}, \\ \Psi_{i+k}^\dagger &=: \exp(-i\bar{e}_i^T \mathcal{K}\Phi) : & \rightarrow & : \exp(i\alpha\bar{\varphi}) : \bar{\chi}_1 = \bar{V}^\dagger \bar{\chi}_1, \\ \Psi_{i+k} &=: \exp(i\bar{e}_i^T \mathcal{K}\Phi) : & \rightarrow & : \exp(-i\alpha\bar{\varphi}) : \bar{\chi}_1^\dagger = \bar{V} \bar{\chi}_{k-1},\end{aligned}\quad (4.4)$$

where $\alpha^2 = (2 + kM)/k$ and $\varphi = \varphi(z)$, $\bar{\varphi} = \bar{\varphi}(\bar{z})$.

There are k Abelian gapping interactions that are expressed by $(k-1)$ vectors Λ_i and by $\bar{\Lambda}$, given in (2.12), (2.13) and (2.14), respectively. The Λ_i are independent of the form of K ; for example:

$$\Lambda_1 = (\Lambda_1^\uparrow, \Lambda_1^\downarrow) = (\underbrace{1, -1, 0, \dots, 0}_k, \underbrace{1, -1, 0, \dots, 0}_k). \quad (4.5)$$

The corresponding interactions have chiral components with vanishing charge. The k -th vector is $\bar{\Lambda}$ in (2.14); for K given in (4.2), it reads:

$$\bar{\Lambda} = (\bar{\Lambda}_\uparrow, \bar{\Lambda}_\downarrow) = (\underbrace{1, 1, \dots, 1}_k, \underbrace{-1, -1, \dots, -1}_k). \quad (4.6)$$

This vector is globally neutral because $Q = \bar{\Lambda}^T \rho = 0$, but its chiral and antichiral components are charged, $\bar{\Lambda}_\uparrow^T t = -\bar{\Lambda}_\downarrow^T t = k$. The corresponding interaction is time-reversal invariant for even k , in agreement with the index $(-1)^R = (-1)^k$ (cf. (1.4) and (2.15)).

The expressions of the non-Abelian interactions are obtained as follows. The U_{Λ_i} are quartic in the fermion fields as in the $k=2$ case, Eq. (3.14), and their projection follows similar steps. After point splitting (3.15) and projection (4.4), one obtains the analogous of (3.17):

$$U_{\Lambda_i} \rightarrow U_\Lambda^{\text{RR}} = \lim_{z_1 \rightarrow z_2} \left[z_{12}^{-2/k} : V^\dagger(z_1) V(z_2) : \chi_1(z_1) \chi_{k-1}(z_2) \right]_{\text{reg.}} \times \left[z \rightarrow \bar{z} \right]. \quad (4.7)$$

These interactions, for $i = 1, \dots, k-1$, are all projected into the same expression U_Λ^{RR} , that involves the identity sectors for both the charged and neutral \mathbb{Z}_k parafermion theories, owing to the fusion rules $V^\dagger \cdot V \sim I$ and $\chi_1 \cdot \chi_{k-1} \sim I$, respectively. The

normal ordering of vertex operators is the same as in the Pfaffian case (3.18), with $\alpha^2 = (2 + kM)/k$. For the parafermions we use the general operator expansion of descendant fields in the identity sector [25],

$$\chi_1(\varepsilon) \chi_{k-1}(0) = \varepsilon^{-2+2/k} + \frac{2h_1}{c_k} \varepsilon^{2/k} T_n(0) + : \chi_1(0) \chi_{k-1}(0) : + \dots, \quad (4.8)$$

where T_n is the stress tensor of the parafermion theory, c_k its central charge and $h_1 = (k-1)/k$ the conformal dimension of χ_1 . Upon combining this two operator expansions, we obtain:

$$U_\Lambda^{RR} = \left(\frac{2h_1}{c_k} T_n + \alpha^2 T_c \right) \left(\frac{2h_1}{c_k} \bar{T}_n + \alpha^2 \bar{T}_c \right). \quad (4.9)$$

This interaction takes the same $T\bar{T}$ form of descendent of the identity already found in the Pfaffian case, and fulfills the same properties. In particular, it cannot provide a mass for neutral excitations.

The projection of the Abelian interaction corresponding to $\bar{\Lambda}$ is slightly more difficult, because it involves $2k$ fermionic fields:

$$\begin{aligned} U_{\bar{\Lambda}} &= : \prod_{i=1}^k \Psi_i^\dagger : : \prod_{i=1}^k \Psi_{k+i} : + \text{h.c.} \\ &= : \prod_{i=1}^k \exp \left(-ie_i \mathcal{K} \Phi(z) \right) : : \prod_{i=1}^k \exp \left(i\bar{e}_i \mathcal{K} \Phi(z) \right) : + \text{h.c.} \end{aligned} \quad (4.10)$$

The operators in each chiral part should be split in k different points $\{z_1, \dots, z_k\}$, with $|z_i - z_j| = \varepsilon \quad \forall i, j$, and later brought back to a common point, $\varepsilon \rightarrow 0$. We use the formula for the normal ordering of k vertex operators [25],

$$\prod_{i=1}^k : \exp(-ie_i \mathcal{K} \Phi(z_i)) : = \prod_{i < j}^k (z_i - z_j)^M : \exp \left[-i \left(\sum_{i=1}^k e_i \right) \mathcal{K} \Phi(z_i) \right] :, \quad (4.11)$$

where the exponent M is given by the K -matrix element. Upon performing the projection (4.4) on individual fermion fields, we re-normal order the k vertex operators V^\dagger , and obtain:

$$U_{\bar{\Lambda}} \rightarrow U_{\bar{\Lambda}}^{RR} = \lim_{\varepsilon \rightarrow 0} \left[\prod_{i < j}^k z_{ij}^{2/k} : (V^\dagger(z))^k : \prod_{i=1}^k \chi_i(z_i) \right] \times \left[z \rightarrow \bar{z} \right] + \text{h.c.} \quad (4.12)$$

The normal ordering of parafermion fields uses the operator product expansions [25],

$$\chi_\ell(z) \chi_{\ell'}(0) \sim z^{-2\ell\ell'/k} \chi_{\ell+\ell'}(0) + \dots, \quad (\ell + \ell' < k), \quad (4.13)$$

$$\chi_\ell(z) \chi_{k-\ell}(0) \sim z^{-2\ell(k-\ell)/k} \left(1 + z^2 \frac{2h_\ell}{c_k} T_n(0) + \dots \right), \quad (4.14)$$

where $h_\ell = \ell(k - \ell)/k$ is the dimension of χ_ℓ . The coincidence limit of the first $(k - 1)$ coordinates $z_i - z = \varepsilon \rightarrow 0$, creates the parafermion field χ_{k-1} with singular behavior given by the sum of exponents in (4.13); for the k -th limit we use (4.14) involving the stress tensor, and obtain,

$$\lim_{\varepsilon \rightarrow 0} \prod_{i=1}^k \chi_i(z_i) = \varepsilon^{1-k} \left(1 + \varepsilon^2 \frac{2h_1}{c_k} T_n(z) \right). \quad (4.15)$$

This singularity exactly cancels that coming from the product of vertex operators in (4.12), leading to final result:

$$U_\Lambda^{RR} =: V^{\dagger k}(z) \bar{V}^k(\bar{z}) + \text{h.c.} \quad (4.16)$$

4.2 Properties of interactions

We now discuss the features of the two edge interactions found for the Read-Rezayi topological insulators, paralleling the analysis of Section 3.3. The time-reversal transformations of Read-Rezayi fields is again inherited from the Abelian fields (2.6) through the projection (4.4), generalizing the result (3.10). It is apparent that the minus sign cannot be assigned to the parafermion field χ_1 , as it would be inconsistent with the fusion rule $(\chi_1)^k \sim I$ for k odd. As already argued, it should be attached to the charged part, leading to the transformations:

$$\begin{aligned} \mathcal{T}: \quad V^\dagger &\rightarrow \bar{V}^\dagger, & \bar{V}^\dagger &\rightarrow -V^\dagger, \\ \chi_i &\rightarrow \bar{\chi}_i, & \bar{\chi}_i &\rightarrow \chi_i, \end{aligned} \quad (4.17)$$

in complete analogy with (3.11). It follows that the neutral interaction $U_\Lambda^{RR} \sim \bar{T}T$ is time-reversal invariant for any k , while the charged interaction U_Λ^{RR} is only invariant for k even, as in the parent Abelian theory. In the even k case, we should again consider the possibility that U_Λ^{RR} breaks spontaneously the symmetry by forcing an expectation value for $V^\dagger \bar{V}$. This cannot happen because $V^\dagger \bar{V}$ is non-local with respect to some excitations of the system. For example, we consider the chiral quasiparticle of smallest charge e^* in (4.3) inserted in the analogous of the correlator (3.23) of the Pfaffian case; one needs the operator expansion of V^\dagger with the quasiparticle vertex operator $V_{e^*} = \exp(i\beta\varphi)$, with $\beta = 1/(2\sqrt{M+1})$, which is $V^\dagger(\varepsilon) V_{e^*}(0) \sim \varepsilon^{-1/\sqrt{2k}}$, leading to a non-trivial monodromy incompatible with spontaneous symmetry breaking.

The properties of interactions should be compared with the analysis based on flux insertions and Kramers theorem [9]. For k odd, this tells that some edge excitations

are protected against any time-reversal invariant interaction and remain massless. Indeed, these are the charged excitations, because the interaction U_{Λ}^{RR} (4.9) is forbidden by time-reversal invariance.

For k even, the Read-Rezayi topological insulators are unstable according to the flux argument. The charged interaction is allowed and gaps all charged excitations, being a cosine of the Abelian field. Regarding the other interaction, $U_{\Lambda}^{RR} \sim \bar{T}T$, it is not sufficient to give mass to the remaining neutral excitations, owing to the arguments discussed in Section 3.3 for $k = 2$.

In order to gap the neutral modes, we consider a quasiparticle interaction that is allowed in the reduced neutral theory where all charged states have acquired very large masses. In this limit, the single-edge partition function of Read-Rezayi topological insulators [9] becomes the following expression:

$$Z^{RR} \rightarrow \sum_{\ell=0, \text{ even}}^k |\chi_0^{\ell}|^2 + \sum_{\ell=0, \ell-\frac{k}{2} \text{ even}}^k \left| \chi_{\frac{k}{2}}^{\ell} \right|^2, \quad (4.18)$$

that contains a subset of the excitations of the Z_k parafermion statistical model [24]. The parafermionic characters χ_m^{ℓ} describe neutral excitations with quantum numbers $(\ell, m) \equiv (\ell, m + 2k) \equiv (k - \ell, m + k)$ and $m = \ell \bmod 2$.

In this theory we introduce the quasiparticle interaction,

$$U_{qp}^{RR} = \bar{\chi}_0^2 \chi_0^2 + \text{h.c.}, \quad (4.19)$$

with dimension $2h_0^2 = 4/(k+2) < 2$, that is relevant and couples to all sectors, since the fusion rules $\chi_0^2 \chi_0^{\ell}$ and $\chi_0^2 \chi_{\frac{k}{2}}^{\ell}$ are different from zero for any allowed value of ℓ [25]. This interaction is generically nonvanishing and gives mass to all the neutral interactions; indeed, such a relevant term coupling to all sectors drives the system into a completely massive phase.

In conclusion, the analysis of interactions in the Read-Rezayi topological insulators confirms the result of the flux argument: some charged excitations remain massless for odd k values, while all excitation become massive for even k .

5 Time-reversal interactions in the non-Abelian Spin-Singlet state

5.1 The NASS state and its parent Abelian state

In this section, we extend our analysis to another prominent non-Abelian state involving spinful chiral electrons, that could be used again to model topological insulators. This provides another test of our approach of projected Abelian states. Some background on the physical motivations and properties of the NASS Hall state can be found in Refs. [22]; we follow the analysis of the quantum numbers, spectrum and partition function of Ref. [24].

In the NASS state, the clustering property of the Read-Rezayi states is extended to spinful electrons, requiring that the (bosonic) ground state wavefunction does not vanish when k electrons with spin up and k with spin down meet at the same point. The conformal field theory description involves generalized parafermions of two kinds, $\chi_{\uparrow 1}$ and $\chi_{\downarrow 1}$, that obey the fusion rules $(\chi_{\uparrow 1})^k = (\chi_{\downarrow 1})^k = I$ and thus reproduce the vanishing property of the wavefunction. These parafermions can be obtained from the coset theory $\widehat{SU(3)_k}/\widehat{U(1)}^2$, to which we should add the $\widehat{U(1)}^2$ symmetry for charge and spin conservation. Although this state involves up and down spin configurations, it is chiral; thus, a time-reversal invariant topological insulator is obtained by adding a specular antichiral state, as in earlier spin-polarized cases. The combined system possesses independent spin and chiral excitations of both kinds, namely chirality and spin are not tied together. Therefore, excitations are always doubled and a single Kramers pair cannot be formed, leading to instability according to the flux insertion argument. The charge parameters are:

$$\nu^\uparrow = \frac{2k}{2kM+3}, \quad e^* = \frac{1}{2kM+3}, \quad (-1)^{2S} = (-1)^{2k} = 1, \quad M \text{ odd}, \quad (5.1)$$

showing that the stability index is always one, as expected. Thus, we should exhibit the gapping interactions consistent with time-reversal symmetry.

The NASS state also possesses a parent Abelian state that describes distinguishable electrons with the same physical features, and that reduces to the non-Abelian state upon projection to identical electrons [22]. Let us recall this Abelian theory in the simplest case $k = 2$ and charge parameter M eventually fixed to $M = 1$, leading to $\nu^\uparrow = 4/7$ and $e^* = 1/7$ in (5.1). For $k = 2$ there are two chiral electrons for each spin orientation, \uparrow, \downarrow , that are distinct by the index $a = 1, 2$, leading to the

four-dimensional chiral K matrix:

$$K = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} + M \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \quad (5.2)$$

The corresponding wavefunctions is:

$$\Psi_{\text{Ab}}^{\text{NASS}} = \prod_{i < j} \left(z_{ij}^{\uparrow} z_{ij}^{\downarrow} w_{ij}^{\uparrow} w_{ij}^{\downarrow} \right)^2 \prod_{i,j} (z_i^{\uparrow} - z_j^{\downarrow})(w_i^{\uparrow} - w_j^{\downarrow}), \quad (5.3)$$

where $z_i^{\uparrow}, z_i^{\downarrow}$ and $w_j^{\uparrow}, w_j^{\downarrow}$ are the coordinates of the two kinds of fermions, with both spin orientations. The overall factor $\prod_{i < j} x_{ij}^M$ has been omitted in $\Psi_{\text{Ab}}^{\text{NASS}}$, where x_i stands for any coordinate type.

The topological insulator is obtained by adding four antichiral components, leading to a eight-dimensional \mathcal{K} matrix in block diagonal form. The elementary fermion fields are vertex operators written in terms of a eight-dimensional scalar field Φ and lattice basis e_i , $i = 1, \dots, 8$ defined in (2.2). From the wavefunction (5.3), the vectors of chiral electrons with up and down spins are identified as follows:

$$\begin{aligned} e_1^{\uparrow} = e_1 &= (1, 0, 0, 0, 0, 0, 0, 0), & e_1^{\downarrow} = e_2 &= (0, 1, 0, 0, 0, 0, 0, 0), \\ e_2^{\uparrow} = e_3 &= (0, 0, 1, 0, 0, 0, 0, 0), & e_2^{\downarrow} = e_4 &= (0, 0, 0, 1, 0, 0, 0, 0), \end{aligned} \quad (5.4)$$

while for antichiral electrons, they read,

$$\begin{aligned} \bar{e}_1^{\downarrow} = e_5 &= (0, 0, 0, 0, 1, 0, 0, 0), & \bar{e}_1^{\uparrow} = e_6 &= (0, 0, 0, 0, 0, 1, 0, 0), \\ \bar{e}_2^{\downarrow} = e_7 &= (0, 0, 0, 0, 0, 0, 1, 0), & \bar{e}_2^{\uparrow} = e_8 &= (0, 0, 0, 0, 0, 0, 0, 1). \end{aligned} \quad (5.5)$$

Owing to the spin assignments in (5.4) and (5.5), we should permute the elements of the basis to match the standard notation of Section two, in which the first (resp. second) half of the components describe spin up (down) electrons, and time-reversal transformations acts by the matrix Σ_1 . The elements of the new basis are ordered as follows:

$$e_1^{\uparrow} \rightarrow e_1, \bar{e}_1^{\uparrow} \rightarrow e_2, e_2^{\uparrow} \rightarrow e_3, \bar{e}_2^{\uparrow} \rightarrow e_4, \bar{e}_1^{\downarrow} \rightarrow e_5, e_1^{\downarrow} \rightarrow e_6, \bar{e}_2^{\downarrow} \rightarrow e_7, e_2^{\downarrow} \rightarrow e_8. \quad (5.6)$$

In the new basis the \mathcal{K} matrix is no longer block-diagonal since the chiralities are

mixed. It reads (hereafter we take $M = 1$):

$$\mathcal{K} = \begin{pmatrix} 3 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & -3 & 0 & -1 & -2 & 0 & -1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & -3 & -1 & 0 & -2 & 0 \\ 0 & -2 & 0 & -1 & -3 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & -1 & 0 & -2 & -1 & 0 & -3 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 3 \end{pmatrix}. \quad (5.7)$$

Note that time-reversal symmetry invariance, $\mathcal{K} = -\Sigma_1 \mathcal{K} \Sigma_1$, is verified. The electron operators and their transformation rule under time reversal are, as usual,

$$\begin{aligned} \mathcal{T}: \quad \Psi_{a\uparrow}^\dagger &= \exp(-ie_{a\uparrow}^T \mathcal{K} \Psi) \rightarrow \bar{\Psi}_{a\downarrow}^\dagger = \exp(-i\bar{e}_{a\downarrow}^T \mathcal{K} \Psi), \quad a = 1, 2, \\ \Psi_{a\downarrow}^\dagger &\rightarrow -\bar{\Psi}_{a\uparrow}^\dagger, \quad \bar{\Psi}_{a\uparrow}^\dagger \rightarrow \Psi_{a\downarrow}^\dagger, \quad \bar{\Psi}_{a\downarrow}^\dagger \rightarrow -\Psi_{a\uparrow}^\dagger. \end{aligned} \quad (5.8)$$

The projection from the Abelian to the NASS theory is realized by identifying electrons of the two species, as follows:

$$\begin{aligned} \Psi_{a\uparrow}^\dagger &\rightarrow \chi_\uparrow : \exp(-i\alpha\varphi_c) \exp(-i\beta\varphi_s) : = \chi_\uparrow V_\alpha^\dagger V_\beta^\dagger, \quad a = 1, 2, \\ \Psi_{a\downarrow}^\dagger &\rightarrow \chi_\downarrow : \exp(-i\alpha\varphi_c) \exp(i\beta\varphi_s) : = \chi_\downarrow V_\alpha^\dagger V_\beta, \\ \bar{\Psi}_{a\downarrow}^\dagger &\rightarrow \bar{\chi}_\downarrow : \exp(i\alpha\bar{\varphi}_c) \exp(-i\beta\bar{\varphi}_s) : = \bar{\chi}_\downarrow \bar{V}_\alpha^\dagger \bar{V}_\beta, \\ \bar{\Psi}_{a\uparrow}^\dagger &\rightarrow \bar{\chi}_\uparrow : \exp(i\alpha\bar{\varphi}_c) \exp(i\beta\bar{\varphi}_s) : = \bar{\chi}_\uparrow \bar{V}_\alpha^\dagger \bar{V}_\beta^\dagger, \end{aligned} \quad (5.9)$$

where φ_c ($\bar{\varphi}_c$) and φ_s ($\bar{\varphi}_s$) are the chiral (antichiral) scalar fields for the charged and spin parts, respectively, and $\chi_\uparrow, \chi_\downarrow$ ($\bar{\chi}_\uparrow, \bar{\chi}_\downarrow$) are the chiral (antichiral) spinful $k = 2$ parafermions with conformal dimension $h = 1/2$, obeying $\chi_\sigma^\dagger = \chi_\sigma$, $\sigma = \uparrow, \downarrow$. The parameters in the vertex operators for charge V_α and spin V_β are $\alpha^2 = 7/4$ and $\beta^2 = 1/4$, respectively.

The time-reversal transformations pass through the projection to the NASS conformal fields; as in earlier cases, we should assign the fermion sign to a conserved quantity of the conformal theory that is related to spin. In the NASS case, the field V_β carries the $U(1)_S$ spin symmetry so the sign can be attached to it as $(-1)^{2S}$. As discussed in [9], spin is not conserved in realistic models of topological insulators, but is broken to spin parity, $U(1)_S \rightarrow \mathbb{Z}_2$, in presence of time-reversal symmetry; thus, the sign is well-defined in general. In conclusion, the time-reversal transformations are:

$$\mathcal{T}: V_\beta^\dagger \rightarrow \bar{V}_\beta, \quad \bar{V}_\beta \rightarrow -V_\beta^\dagger, \quad (5.10)$$

while V_α and the χ_σ are only conjugated.

5.2 Projected interactions

In the new basis (5.6), the vectors specifying the time-reversal invariant interactions of the parent Abelian theory (2.12)-(2.13) and (2.14) have the form, according to the discussion in Section two,

$$\begin{aligned}\Lambda_1 &= (1, -1, 0, 0, 1, -1, 0, 0), & \Lambda_2 &= (1, 0, -1, 0, 1, 0, -1, 0), \\ \Lambda_3 &= (1, 0, 0, -1, 1, 0, 0, -1), & \bar{\Lambda} &= (1, -1, 1, -1, -1, 1, -1, 1).\end{aligned}\quad (5.11)$$

We recall that the first three obey $\Sigma_1 \Lambda_i = \Lambda_i$, $i = 1, 2, 3$, while the forth fulfills $\Sigma_1 \bar{\Lambda} = -\bar{\Lambda}$, and is time-reversal invariant owing to $\bar{\Lambda}^T \Sigma_{\downarrow} \rho = 0$. Using the general formula for vertex operators (2.2) and keeping in mind the identifications of the basis (5.6), we obtain the expressions:

$$\begin{aligned}U_{\Lambda_1} &=: \Psi_{1\uparrow}^\dagger \bar{\Psi}_{1\uparrow} \bar{\Psi}_{1\downarrow}^\dagger \Psi_{1\downarrow} : + \text{h.c.}, \\ U_{\Lambda_2} &=: \Psi_{1\uparrow}^\dagger \Psi_{2\uparrow} \bar{\Psi}_{1\downarrow}^\dagger \bar{\Psi}_{2\downarrow} : + \text{h.c.}, \\ U_{\Lambda_3} &=: \Psi_{1\uparrow}^\dagger \bar{\Psi}_{2\uparrow} \bar{\Psi}_{1\downarrow}^\dagger \Psi_{2\downarrow} : + \text{h.c.}, \\ U_{\bar{\Lambda}} &=: \Psi_{1\uparrow}^\dagger \bar{\Psi}_{1\uparrow} \Psi_{2\uparrow}^\dagger \bar{\Psi}_{2\uparrow} \bar{\Psi}_{1\downarrow}^\dagger \Psi_{1\downarrow} \bar{\Psi}_{2\downarrow}^\dagger \Psi_{2\downarrow} : + \text{h.c.}\end{aligned}\quad (5.12)$$

We now compute the projected forms of each interaction by the method of point splitting and re-normal ordering already applied before. The operator product expansions of Abelian electrons needed in the calculations can be read from the wavefunction (5.3):

$$\begin{aligned}\Psi_{a\uparrow}^\dagger(z) \Psi_{a\downarrow}^\dagger(0) &= z^2 : \Psi_{a\uparrow}^\dagger \Psi_{a\downarrow}^\dagger :, & a &= 1, 2, \\ \Psi_{1\sigma}^\dagger(z) \Psi_{2\sigma'}^\dagger(0) &= z : \Psi_{1\sigma}^\dagger \Psi_{2\sigma'}^\dagger :, & \sigma, \sigma' &= \uparrow, \downarrow,\end{aligned}\quad (5.13)$$

while those of the vertex operators V_α and V_β are expressed by the charges α and β defined before. We also need the following operator products of parafermions χ_σ , $\sigma = \uparrow, \downarrow$ [22],

$$\chi_\sigma(z) \chi_\sigma(0) = z^{-1} I + \lambda z T_n(0) + \dots, \quad \sigma = \uparrow, \downarrow, \quad (5.14)$$

$$\chi_\uparrow(z) \chi_\downarrow(0) = z^{-1/2} \chi_{\uparrow\downarrow}(0) - z \chi_\uparrow \partial \chi_\downarrow(0) + \dots, \quad (5.15)$$

$$\chi_{\uparrow\downarrow}(z) \chi_{\uparrow\downarrow}(0) = z^{-1} I + \dots, \quad (5.16)$$

$$\chi_{\uparrow\downarrow}(z) \chi_\uparrow(0) = z^{-1/2} \chi_\downarrow(0) + \dots, \quad (5.17)$$

$$\chi_{\uparrow\downarrow}(z) \chi_\downarrow(0) = z^{-1/2} \chi_\uparrow(0) + \dots, \quad (5.18)$$

where λ is a constant, $\chi_{\uparrow\downarrow}$ is another parafermionic field with conformal dimension $h = 1/2$, and T_n is the stress tensor of the parafermion theory.

Let us first consider the interaction U_{Λ_1} in (5.12). Upon splitting the two chiral fermions in points z_1 and z_2 , performing the projection and re-normal ordering the V fields, we obtain:

$$\begin{aligned}
U_{\Lambda_1} &\rightarrow U_{\Lambda_1}^{NASS} \\
&= \left[z_{12}^{1/2} : V_{\alpha}^{\dagger} V_a :: V_{\beta}^{\dagger} V_{\beta} : \left(z_{12}^{-1/2} \chi_{\uparrow\downarrow} + O(z_{12}) \right) \right] \times \left[z \rightarrow \bar{z}, \uparrow \rightarrow \downarrow \right] + \text{h.c.} \\
&= : V_{\beta}^{\dagger 2} \bar{V}_{\beta} : \chi_{\uparrow\downarrow} \bar{\chi}_{\uparrow\downarrow} + \text{h.c.}, \tag{5.19}
\end{aligned}$$

showing that this interaction involves the spin sector of the theory.

Next we observe that the interactions U_{Λ_3} and U_{Λ_1} differ in the type $a = 1, 2$ of some fields, that is irrelevant after projection to the NASS theory. Nevertheless, a different singularity of original Abelian fields, namely $\Psi_{1\uparrow}^{\dagger} \Psi_{1\downarrow} \sim z^{-2}$ versus $\Psi_{1\uparrow}^{\dagger} \Psi_{2\downarrow} \sim z^{-1}$ (cf. (5.3)), implies a slightly different result in the normal-ordering procedure :

$$U_{\Lambda_3}^{NASS} = (\alpha \partial \varphi_c - \beta \varphi_s) (\alpha \partial \bar{\varphi}_c - \beta \bar{\varphi}_s) : V_{\beta}^{\dagger 2} \bar{V}_{\beta} : \chi_{\uparrow\downarrow} \bar{\chi}_{\uparrow\downarrow} + \text{h.c.} \tag{5.20}$$

Therefore, $U_{\Lambda_3}^{NASS}$ differs from $U_{\Lambda_1}^{NASS}$ for the presence of descendant fields in the same conformal sector. Although their explicit expressions are different, we should not consider the two interactions as independent, because they are equivalent in the ability of gapping excitations, as discussed in Section 3.2.

The analysis of U_{Λ_2} follows similar steps and we find:

$$\begin{aligned}
U_{\Lambda_2} &\rightarrow U_{\Lambda_2}^{NASS} \\
&= \left[z_{12}^{-1} : V_{\alpha}^{\dagger} V_a :: V_{\beta}^{\dagger} V_{\beta} : (z_{12}^{-1} + \lambda z_{12} T_{pf}(z_2)) \right] \times \left[z \rightarrow \bar{z}, \uparrow \rightarrow \downarrow \right] + \text{h.c.} \\
&= (\alpha^2 T_c + \beta^2 T_s + \lambda T_{pf} - \alpha \beta \partial \varphi_c \partial \varphi_s) \\
&\quad \times (\alpha^2 \bar{T}_c + \beta^2 \bar{T}_s + \lambda \bar{T}_{pf} - \alpha \beta \bar{\partial} \bar{\varphi}_c \bar{\partial} \bar{\varphi}_s), \tag{5.21}
\end{aligned}$$

where $T_s = -1/2(\partial \varphi_s)^2$ and $T_c = -1/2(\partial \varphi_c)^2$. It is apparent that this interaction involves descendant fields in the identity sectors of charge, spin and parafermionic parts of the conformal theory.

We finally consider the projection of the interaction $U_{\bar{\Lambda}}$ in (5.12). In this case, we should separate four chiral Abelian fields at points $\{z_1, z_2, z_3, z_4\}$, with $|z_i - z_j| = \varepsilon$, and then bring them back together after projection. Focusing on the chiral part, and using (5.14) for fusing parafermions, we obtain:

$$\begin{aligned}
U_{\bar{\Lambda}} \Big|_z &= \lim_{\varepsilon \rightarrow 0} (z_{12} z_{13}^2 z_{14} z_{23} z_{24}^2 z_{34})^{-1} \Psi_{1\uparrow}^{\dagger}(z_1) \Psi_{2\uparrow}^{\dagger}(z_2) \Psi_{1\downarrow}^{\dagger}(z_3) \Psi_{2\downarrow}^{\dagger}(z_4) \\
&\rightarrow \lim_{\varepsilon \rightarrow 0} z_{12} z_{13}^{-1/2} z_{14}^{1/2} z_{23}^{1/2} z_{24}^{-1/2} z_{34} : V_{\alpha}^{\dagger 4}(z) : \chi_{\uparrow}(z_1) \chi_{\uparrow}(z_2) \chi_{\downarrow}(z_3) \chi_{\downarrow}(z_4) \\
&= \lim_{\varepsilon \rightarrow 0} \left(\frac{z_{14} z_{23}}{z_{13} z_{24}} \right)^{\frac{1}{2}} : V_{\alpha}^{\dagger 4}(z) : . \tag{5.22}
\end{aligned}$$

Therefore, the forth interaction is:

$$U_{\Lambda}^{NASS} =: V_{\alpha}^{\dagger 4} \bar{V}_{\alpha}^4 : + \text{h.c.} \quad (5.23)$$

5.3 Properties of NASS interactions

Let us summarize the three independent time-reversal invariant electron interactions that have been obtained in the NASS state with $k = 2$:

$$U_{\Lambda_2}^{NASS} = (\alpha^2 T_c + \beta^2 T_s + \lambda T_{pf} - \alpha \beta \partial \varphi_c \partial \varphi_s) \times (\alpha^2 \bar{T}_c + \beta^2 \bar{T}_s + \lambda \bar{T}_{pf} - \alpha \beta \bar{\partial} \bar{\varphi}_c \bar{\partial} \bar{\varphi}_s), \quad (5.24)$$

$$U_{\Lambda_1}^{NASS} = : V_{\beta}^{\dagger 2} \bar{V}_{\beta}^2 : \chi_{\uparrow\downarrow} \bar{\chi}_{\uparrow\downarrow} + \text{h.c.}, \quad (5.25)$$

$$U_{\Lambda}^{NASS} = : V_{\alpha}^{\dagger 4} \bar{V}_{\alpha}^4 : + \text{h.c.} \quad (5.26)$$

These interactions involve neutral, spinful and charged operators within each chirality, respectively. As in previous theories, we can argue that spontaneous breaking of time-reversal symmetry cannot be induced by $U_{\Lambda_1}^{NASS}$, because the non-invariant operator $V_{\beta}^{\dagger} \bar{V}_{\beta}$ cannot acquire an expectation value, being nonlocal with respect to some chiral excitations. Therefore, the two interactions $U_{\Lambda_1}^{NASS}$ and U_{Λ}^{NASS} are of Abelian type and provide a mass to all charged and spinful excitations.

Regarding the neutral interaction $U_{\Lambda_2}^{NASS}$, this involves derivative operators and does not gap the neutral excitations, owing to the argument of Section 3.3. Again, we consider the reduced theory where all charged and spinful excitations have decoupled: in this limit, the NASS partition function [9] reduces to the following expression:

$$Z^{NASS} \rightarrow |I|^2 + |\rho|^2 + |\psi_{12}|^2 + |\sigma_3|^2. \quad (5.27)$$

This tells us the remaining neutral quasiparticles: the ρ , ψ_{12} and σ_3 excitations of the $SU(3)$ parafermion theory [22]. In this theory, the quasiparticle interaction,

$$U_{qp}^{NASS} = \bar{\rho} \rho, \quad (5.28)$$

is relevant, $2h_{\rho} = 6/5 < 2$, and couples to all neutral excitations in (5.27) [22]. Therefore, this interaction is generically present and drives the system into a completely massive phase.

In conclusion, we found the needed interactions that confirm the instability of this system as predicted by the flux-insertion argument.

6 Conclusions

In this paper, we have found the edge interactions that can gap time-reversal invariant topological insulators made by chiral-antichiral pairs of non-Abelian Hall states. In the cases where the flux-insertion argument predicted the instability of the systems, we find a sufficient set of interactions that actually let them decay. In case of stability, as e.g. the \mathbb{Z}_k parafermionic Read-Rezayi states with k odd, we found instead that the available interactions are not sufficient to gap the system completely. These results complement the stability analysis of our previous paper [9].

That unstable topological insulators possess enough interactions for decaying could be considered as a natural result. However, in checking this property explicitly we found some interesting features of these models. Firstly, we noticed that the spin sign in time-reversal transformations is carried by Abelian conformal fields that can represent faithfully the spin parity $(-1)^{2S}$. Secondly, we observed that the expressions of interactions require normal ordering and that this procedure is free from ambiguities due to the electron field obeying Abelian fusion rules (a so-called simple current) [24]. Thirdly, we found that neutral excitations are gapped by a quasiparticle interaction that is allowed and generically present in the low-energy limit of the theory. The physical origin of this interaction remains to be understood, since it does not correspond to electron scattering at impurities.

Our study of interaction was based on previous analyses of Abelian theories combined with particular projections that relate Abelian and non-Abelian Hall states [29] [22]. It is likely that this kind of approach could be extended to other non-Abelian states that are described by the more general Wen's parton construction also involving Abelian states [10] [32].

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References

- [1] X. L. Qi, S. C. Zhang, “Topological insulators and superconductors” *Rev. Mod. Phys.* **83** (2011) 1057; E. Fradkin, *Field Theories of Condensed Matter Systems*, 2nd edition, Cambridge Univ. Press (2013), Cambridge UK.
- [2] M. König, S. Wiedmann, C. Bruene, A. Roth, H. Buhmann, L. W. Molenkamp, X. L. Qi, S. C. Zhang, “Quantum Spin Hall Insulator State in HgTe Quantum Wells”, *Science* **318** (2007) 766; M. Koenig, H. Buhmann, L. W. Molenkamp, T. Hughes, C. -X. Liu, X. -L. Qi, S. C. Zhang, “The Quantum Spin Hall Effect: Theory and Experiment”, *J. Phys. Soc. Jpn.* **77** (2008) 031007.
- [3] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, M. Z. Hasan, “A topological Dirac insulator in a quantum spin Hall phase”, *Nature* **452** (2008) 970; Y. Xia, D. Qian, D. Hsieh, L. Wray, A. Pal, H. Lin, A. Bansil, D. Grauer, Y. S. Hor, R. J. Cava, M. Z. Hasan, “Observation of a large-gap topological-insulator class with a single Dirac cone on the surface”, *Nature Physics* **5** (2009) 398; Y. L. Chen, J. G. Analytis, J.-H. Chu, Z. K. Liu, S.-K. Mo, X. L. Qi, H. J. Zhang, D. H. Lu, X. Dai, Z. Fang, S. Zhang, I. R. Fisher, Z. Hussain, Z.-X. Shen, “Experimental Realization of a Three-Dimensional Topological Insulator Bi_2Te_3 ”, *Science* **325** (2009) 178.
- [4] M. A. Metlitski, C. L. Kane, M. P. A. Fisher, “A symmetry-respecting topologically-ordered surface phase of 3d electron topological insulators”, preprint arXiv:1306.3286; X. Chen, L. Fidkowski, A. Vishwanath, “Symmetry Enforced Non-Abelian Topological Order at the Surface of a Topological Insulator”, *Phys. Rev. B* **89** (2014) 165132; P. Bonderson, C. Nayak, X.-L. Qi, “A Time-Reversal Invariant Topological Phase at the Surface of a 3D Topological Insulator”, *J. Stat. Mech.* (2013) P09016.
- [5] M. Levin, A. Stern, “Fractional Topological Insulator”, *Phys. Rev. Lett.* **103** (2009) 196803, “Classification and analysis of two dimensional abelian fractional topological insulators,” *Phys. Rev. B* **86** (2012) 115131.
- [6] X. Chen, Z. X. Liu, X. G. Wen, “Two-dimensional symmetry-protected topological orders and their protected gapless edge excitation”, *Phys. Rev. B* **23** (2011) 235141.

- [7] T. Neupert, L. Santos, S. Ryu, C. Chamon, C. Mudry, “Fractional topological liquids with time-reversal symmetry and their lattice realization”, *Phys. Rev. B* **84** (2011) 165107; “Time-reversal symmetric hierarchy of fractional incompressible liquids”, *Phys. Rev. B* **84** (2011) 165138.
- [8] Y. M. Lu, A. Vishwanath, “Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach”, *Phys. Rev. B* **86** (2012) 125119.
- [9] A. Cappelli, E. Randellini, “Partition Functions and Stability Criteria of Topological Insulators”, *JHEP* **12** (2013) 101.
- [10] Z. Ringel, A. Stern “The \mathbb{Z}_2 -anomaly and boundaries of topological insulators”, *Phys. Rev. B* **88** (2013) 115307; M Koch-Janusz, Z. Ringel, “Interacting and fractional topological insulators via the \mathbb{Z}_2 chiral anomaly”, *Phys. Rev. B* **89** (2014) 075137.
- [11] O. M. Sule, X. Chen, S. Ryu, “Symmetry-protected topological phases and orbifolds”, *Phys. Rev. B* **88** (2013) 075125; C.T. Hsieh, O. M. Sule, G. Y. Cho, S. Ryu, R. G. Leigh, “Symmetry-protected Topological Phases, Generalized Laughlin Argument and Orientifolds”, preprint arXiv:1403.6902; T. C. Hsieh, T. Morimoto, S. Ryu, “CPT theorem and classification of topological insulators and superconductors”, preprint arXiv:1406.0307.
- [12] M. Levin, Z. C. Gu “Braiding statistics approach to symmetry-protected topological phases”, *Phys. Rev. B* **86** (2012) 115109; M. Levin, “Protected edge modes without symmetry”, *Phys. Rev. X* **3** (2013) 021009; Z. C. Gu, M. Levin, “The effect of interactions on 2D fermionic symmetry-protected topological phases with \mathbb{Z}_2 symmetry”, *Phys. Rev. B* **89** (2014) 201113.
- [13] M.A. Metlitski, L. Fidkowski, X. Chen, A. Vishwanath, “Interaction effects on 3D topological superconductors: surface topological order from vortex condensation, the 16 fold way and fermionic Kramers doublets”, preprint arXiv:1406.3032.
- [14] C. L. Kane, E. J. Mele, “ \mathbb{Z}_2 Topological Order and the Quantum Spin Hall Effect,” *Phys. Rev. Lett.* **95** (2005) 146802; L. Fu, C. L. Kane, “Time reversal polarization and a \mathbb{Z}_2 adiabatic spin pump”, *Phys. Rev. B* **74** (2006) 195312; “Topological insulators with inversion symmetry”, *Phys. Rev. B* **76** (2007) 045302; L. Fu, C. L. Kane, E. J. Mele, “Topological Insulators in Three Dimensions”, *Phys. Rev. Lett.* **98** (2007) 106803.

- [15] J. E. Moore, L. Balents, “Topological invariants of time-reversal-invariant band structures”, Phys. Rev. B **75** (2007) 121306; R. Roy, “ Z_2 classification of quantum spin Hall systems: An approach using time-reversal invariance”, Phys. Rev. B **79** (2009) 195321.
- [16] X. -L. Qi, S. -C. Zhang, “Spin Charge Separation in the Quantum Spin Hall State,” Phys. Rev. Lett. **101** (2008) 086802.
- [17] R. B. Laughlin, “Quantized Hall conductivity in two-dimensions,” Phys. Rev. B **23** (1981) 5632.
- [18] A. Cappelli, G. V. Dunne, C. A. Trugenberger, G. R. Zemba, “Conformal symmetry and universal properties of quantum Hall states,” Nucl. Phys. B **398** (1993) 531.
- [19] X. G. Wen, Quantum Field Theory of Many-body Systems, Oxford Univ. Press (2007), Oxford.
- [20] G. W. Moore, N. Read, “Nonabelions in the fractional quantum hall effect,” Nucl. Phys. B **360** (1991) 362.
- [21] N. Read, E. Rezayi, “Beyond paired quantum Hall states: Parafermions and incompressible states in the first excited Landau level,” Phys. Rev. B **59** (1999) 8084.
- [22] E. Ardonne, K. Schoutens, “New Class of Non-Abelian Spin-Singlet Quantum Hall States”, Phys. Rev. Lett. **82** (1999) 5096; E. Ardonne, N. Read, E. Rezayi, K. Schoutens, “Non-abelian spin-singlet quantum Hall states: wavefunctions and quasihole state counting”, Nucl. Phys. B **607** (2001) 549; K. Schoutens, E. Ardonne, & F. J. M. van Lankvelt, “Paired and Clustered Quantum Hall States”, in Proc. of the NATO Advanced Research Workshop ”Statistical Field Theories” Como (Italy), June 18-23, 2001, A. Cappelli, G. Mussardo, eds. (Kluwer Academic Publishers 2002); E. Ardonne, K. Schoutens, “Wavefunctions for topological quantum registers”, Ann. Phys. **322**, (2007) 201.
- [23] A. Cappelli and G. R. Zemba, “Modular invariant partition functions in the quantum Hall effect,” Nucl. Phys. B **490** (1997) 595.
- [24] A. Cappelli, G. Viola, “Partition Functions of Non-Abelian Quantum Hall States,” J. Phys. A **44** (2011) 075401.

- [25] P. Di Francesco, P. Mathieu, D. Sénéchal, Conformal Field Theory, Springer-Verlag (1997) New York.
- [26] E. Witten, “An $SU(2)$ anomaly” Phys. Lett. B **117** (1982) 324; A. J. Niemi, G. W. Semenoff, “Axial Anomaly Induced Fermion Fractionization and Effective Gauge Theory Actions in Odd Dimensional Space-Times,” Phys. Rev. Lett. **51**, 2077 (1983); A. N. Redlich, “Parity Violation and Gauge Noninvariance of the Effective Gauge Field Action in Three-Dimensions” Phys. Rev. D **29** (1984) 2366; S. Deser, L. Griguolo, D. Seminara, “Effective QED actions: Representations, gauge invariance, anomalies and mass expansions,” Phys. Rev. D **57** (1998) 7444.
- [27] S. Ryu, S. C. Zhang, “Interacting topological phases and modular invariance,” Phys. Rev. B **85** (2012) 245132.
- [28] A. Cappelli, L. S. Georgiev, I. T. Todorov, “A Unified conformal field theory description of paired quantum Hall states,” Commun. Math. Phys. **205** (1999) 657.
- [29] A. Cappelli, L. S. Georgiev, I. T. Todorov, “Parafermion Hall states from coset projections of Abelian conformal theories,” Nucl. Phys. B **599** (2001) 499.
- [30] F. D. M. Haldane, “Stability of Chiral Luttinger Liquids and Abelian Quantum Hall States”, Phys. Rev. Lett. **74** (1995) 2090.
- [31] Al. B. Zamolodchikov, “From tricritical Ising to critical Ising by thermodynamic Bethe ansatz”, Nucl. Phys. B **358** (1991) 524; A. B. Zamolodchikov, “Expectation value of composite field $T\bar{T}$ in two-dimensional quantum field theory”, preprint arXiv:hep-th/0401146; M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo, “Quantisation of the effective string with TBA”, preprint arXiv:1305.1278.
- [32] M. Barkeshli, X.G. Wen, “Effective field theory and projective construction for Z_k parafermion fractional quantum Hall states”, Phys. Rev. B **81** (2010) 155302.